FLUTTER OF A DEFORMABLE ROCKET IN SUPERSONIC FLOW

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Summary—The equation of elastic vibration of a slender body in a supersonic flow is derived, damping and axial forces being taken into consideration.

A general solution method is proposed, consisting in reducing the original equation to a Volterra integral equation applying the method of initial parameters. An approximate solution method is also given. Results of two computation examples of the flutter velocity are given.

1. *Introduction*

The object of the present paper is to establish linearized equations of supersonic flow past a deformable slender body with tail and to propose methods for computing flutter velocity. A preliminary statement of this problem was given in[1]. The aerodynamic considerations are based on the Ref.[2].

The equation of lateral vibration of the rocket in supersonic flow obtained here has the character of the equation of lateral vibration of a beam with variable cross-section, with the co-action of axial forces, internal damping and aerodynamic forces (described by terms of the type of Coriolis forces, centrifugal forces and the associated air mass).

A general method for integrating the equation of the problem is proposed. This consists in applying the method of initial parameters and reducing the original equation to a Volterra integral equation. Then the critical parameters are determined. Next, approximate solution methods are given, as well as examples of determining the critical velocity of flutter.

As a result it is shown that if the rocket is designed as an elastically deformable body, the flutter phenomenon appears. It is shown in addition that the flutter velocities lie within the limits of flight velocities for average rockets.

2. *Equation of Vibration of a Slender Body in a Linearized Supersonic Flow*

The problem of determining the aerodynamic forces acting on a slender body in a non-steady state flow was dealt with by many authors. In the present paper the relations given by J. W. Miles[2] are used, he showed that in the case of a linearized potential flow the transversal force acting per unit length of the body may be expressed by the equation (Fig. 1)

$$\frac{\partial F}{\partial x} = \varrho_0 U^2 \frac{DM}{Dt} \tag{2.1}$$

where ϱ_0 is the density of the air, U the air velocity and

$$\frac{D}{Dt} = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

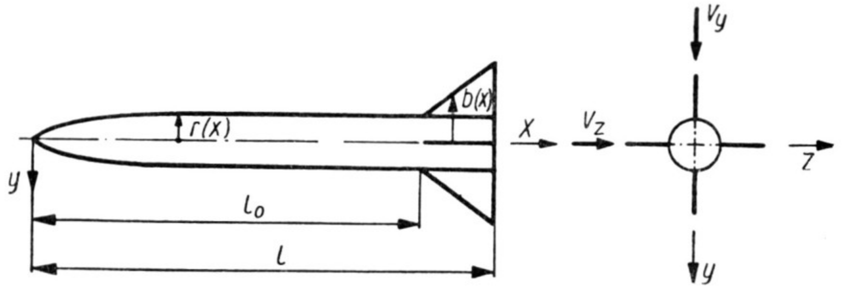

FIG. 1.

For the segment of the rocket without fins

$$M = \pi r^2 (v_y + i v_z) \tag{2.2}$$

and for the finned segment

$$M = \pi\left(b^2 - r^2 + \frac{r^4}{b^2}\right) v_y + i\pi\left(b^2 - r^2 + \frac{r^4}{b^2}\right) v_z \tag{2.3}$$

By assuming that the motion takes place in the xy-plane only, we obtain

$$\frac{\partial F}{\partial x} = \varrho_0 U^2 v \frac{\mathrm{d}A}{\mathrm{d}x} + \varrho_0 U^2 A\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}\right) \tag{2.4}$$

where for the segment without fins

$$A = \pi r^2 \tag{2.5'}$$

for finned segment

$$A = \pi\left(b^2 - r^2 + \frac{r^4}{b^2}\right) \tag{2.5''}$$

and

$$-v = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} \tag{2.6}$$

Quantities x and t are dimensionless related to l and l/U respectively.

Substituting (2.6) in (2.4) and returning to quantities with dimensions, we obtain*

$$-\frac{\partial F}{\partial x}=\varrho_0 U^2 A\frac{\partial^2 y}{\partial x^2}+2\varrho_0 UA\frac{\partial^2 y}{\partial x\,\partial t}+\varrho_0 A\frac{\partial^2 y}{\partial t^2}+\varrho_0 U^2\frac{\mathrm{d}A}{\mathrm{d}x}\frac{\partial y}{\partial x}+\varrho_0 U\frac{\mathrm{d}A}{\mathrm{d}x}\frac{\partial y}{\partial t} \tag{2.7}$$

The equation of transverse vibration of the rocket has the form

$$\frac{\partial^2}{\partial x_2}\left[EJ\frac{\partial^2 y}{\partial x^2}+\eta\frac{\partial^3 y}{\partial x^2\,\partial t}\right]+\varrho^*(x)A\frac{\partial^2 y}{\partial t^2}+\frac{\partial}{\partial x}\left[P(x,t)\frac{\partial y}{\partial x}\right]-\frac{\partial F}{\partial x}=0 \tag{2.8}$$

where $\varrho^*(x)$ is the density of the rocket, $P(x, t)$ the axial force and η the damping coefficient or, after substituting (2.7)

$$\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 y}{\partial x^2}+\eta\frac{\partial^3 y}{\partial x^2\partial t}\right)+\varrho_0 U^2 A\frac{\partial^2 y}{\partial x^2}+\frac{\partial}{\partial x}\left[P(x,t)\frac{\partial y}{\partial x}\right]+$$
$$+2\varrho_0 UA\frac{\partial^2 y}{\partial x\partial t}+\varrho_0 U^2\frac{\mathrm{d}A}{\mathrm{d}x}\frac{\partial y}{\partial x}+\varrho_0 U\frac{\mathrm{d}A}{\mathrm{d}x}\frac{\partial y}{\partial t}+[\varrho_0+\varrho^*(x)]\,A\frac{\partial^2 y}{\partial t^2}=0. \tag{2.9}$$

with the cross section A expressed according to (2.5).

In the following we shall assume that $P(x, t)=0$, although there is no essential difficulty in taking the axial force into consideration.

The boundary conditions are

$$y''(0)=y'''(0)=y''(l)=y'''(l) \tag{2.10}$$

The presence of fins or a wing introduces a discontinuity of the rocket cross-section and its derivative with respect to x which seriously complicates numerical computations in some cases.

In practical computations the fin effect may be expressed in the form of a force applied at the centre of pressure of tail and expressed by the equation

$$F_t=a_1\left(\frac{\partial y}{\partial t}\right)_{x=x_0}+a_2\left(\frac{\partial^2 y}{\partial t^2}\right)_{x=x_0}+a_3\left(\frac{\partial y}{\partial x}\right)_{x=x_0}+$$
$$+a_4\left(\frac{\partial^2 y}{\partial x\partial t}\right)_{x=x_0}+a_5\left(\frac{\partial^3 y}{\partial x\partial t^2}\right)_{x=x_0} \tag{2.11}$$

where

$$a_1=-\varrho_0 U\int_{l_1}^{l_2}\frac{\mathrm{d}A_1}{\mathrm{d}x}\,\mathrm{d}x$$

$$a^2=-\varrho_0 U^2\int_{l_1}^{l_2}A_1\,\mathrm{d}x$$

* The same result for the non-finned segment may be obtained by the method given in Ref. (1).

$$a_3 = -\varrho_0 U^2 \int\limits_{l_1}^{l_2} \frac{\mathrm{d}A_1}{\mathrm{d}x}\,\mathrm{d}x$$

$$a_4 = -2\varrho_0 U \int\limits_{l_1}^{l_2} A_1\,\mathrm{d}x + \varrho_0 U \int\limits_{l_1}^{l_2} \frac{\mathrm{d}A_1}{\mathrm{d}x}(x_0 - x)\,\mathrm{d}x$$

$$a_5 = -\varrho_0 \int\limits_{l_1}^{l_2} A_1 (x_0 - x)\,\mathrm{d}x \tag{2.12}$$

$$A_1 = \pi\left[b^2(x) - r^2 + \frac{r^4}{b^2(x)}\right] - \pi r^2 = \pi\left[b^2(x) - 2r^2 + \frac{r^4}{b^2(x)}\right] \tag{2.13}$$

it being assumed that on the finned segment the radius of the fuselage is constant ($r = \text{const}$), x_0 = the coordinate of the centre of pressure on the fins, l_1 and l_2 = the projection of the force and aft end of the tail on the x-axis, respectively. Hence, the simplified equation of lateral vibration takes the form

$$\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 y}{\partial x_2} + \eta\frac{\partial^3 y}{\partial x^2 \partial t}\right) + \varrho_0 U^2 A \frac{\partial^2 y}{\partial x^2} +$$

$$+ 2\varrho_0 U A \frac{\partial^2 y}{\partial x \partial t} + \varrho_0 U^2 \frac{\mathrm{d}A}{\mathrm{d}x}\frac{\partial y}{\partial x} + \varrho_0 U \frac{\mathrm{d}A}{\mathrm{d}x}\frac{\partial y}{\partial t} +$$

$$+ [\varrho_0 + \varrho^*(x)]\, A \frac{\partial^2 y}{\partial t^2} - \delta(x - x_0) F_t = 0 \tag{2.14}$$

where A is expressed according to the Eq. (2.5′) for the entire length of the rocket.

3. *General Solution of the Problem*

Let us consider now the general accurate method for constructing the solution of the problem just stated. We shall start from the Eq. (2.9) and then we shall show the application of this method to the solution of the Eq. (2.14).

The Eq. (2.9) should, in principle, be replaced by two, as the mathematical expression of the coefficients of the equation changes considerably if we pass to the finned segment. However, from the point of view of computation it is convenient to replace the coefficients with a single expression for the entire length of the rocket, by means of approximate functions.

Then, if the solution of the Eq. (2.9) is assumed in the form

$$y(x, t) = e^{i\omega t} X(x) \tag{3.1}$$

it will become

$$[EJX''(1+i\eta\omega)]''+(P+\varrho_0U^2A)X''+(2\varrho_0UAi\omega+ \\ +\varrho_0U^2A'+P')X'+[\varrho_0UA'i\omega-(\varrho_0+\varrho^*(x))A\omega^2]X=0 \quad (3.2)$$

This equation may be written thus

$$X^{\mathrm{IV}}+a_1(x)X'''+a_2(x)X''+a_3(x)X'+a_4(x)X=0 \quad (3.3)$$

where the coefficients $a_i(x)$ will be obtained by confrontation with the Eq. (3.2).

The solution of the problem will be realized thus. We shall reduce our equation to a Volterra integral equation of the 2-nd kind. This will be done in two ways. In the first, general, arbitrary continuous coefficients $a_i(x)$ will be assumed.

In the second the coefficients will be assumed in the form of polynomials of an arbitrary degree which enables, by applying the Laplace integral transformation, the obtainment of a particularly simple kernel.

The problem is treated as an initial parameter problem in relation to x, assuming X, X', X'', X''', to be known for $x=0$. Two of these parameters are really known $[X''(0)=X'''(0)=0]$, the other two are not. Solving the Volterra integral equation by means of the resolving kernel and arranging the solution in terms of the remaining two constants we make use for their determination of the remaining two boundary conditions for $x=l$.

Setting equal to zero the determinant of the system of equations for the two remaining initial parameters we obtain the characteristic equation.

Assuming ω in the complex form

$$\omega=\alpha+i\beta \quad (3.4)$$

we seek for the equation for $\beta=0$ and determine, from the principle of argument the flutter velocity.

With such a statement of the problem the boundary conditions for $x=l$ are such

$$X''(l)=X'''(l)=0 \quad (3.5)$$

We proceed now to reduce the Eq. (3.3) to a Volterra integral equation. Denoting the initial values by

$$\begin{aligned} X(0)&=A\\ X'(0)&=B\\ X''(0)&=X'''(0)=0 \end{aligned} \quad (3.6)$$

the Eq. (3.3) may be reduced to the following Volterra integral equation

$$\varphi(x)+\int_0^x K(x,\xi)\varphi(\xi)\mathrm{d}\xi=f(x) \quad (3.7)$$

where

$$D^n X = \frac{d^4 X}{dx^4} = \varphi(x) \tag{3.8}$$

Hence

$$D^{-1}\varphi = \int_0^x \varphi(\xi)\,d\xi$$

$$\cdots\cdots\cdots\cdots$$

$$D^{-4}\varphi = \int_0^x \frac{(x-\xi)^3}{3!}\varphi(\xi)\,d\xi \tag{3.9}$$

therefore

$$\frac{d^3X}{dx^3} = D^{-1}\varphi; \qquad \frac{d^2X}{dx^2} = D^{-2}\varphi$$

$$\frac{dX}{dx} = B + D^{-3}\varphi; \qquad X = A + B_x + D^{-4}\varphi \tag{3.10}$$

Substituting (3.10) in the Eq. (3.3), use being made of (3.9) and (3.6) we obtain for the kernel $K(x, \xi)$ and $f(x)$ the following expressions

$$K(x, \xi) = \sum_{i=1}^{4} a_i(x) \frac{(x-\xi)^{i-1}}{(i-1)!} \tag{3.11}$$

$$f(x) = (A+Bx)a_4(x) + Aa_3(x) \tag{3.12}$$

The function $X(x)$ will be obtained from (3.10).

The Eq. (3.3) may be reduced to a Volterra equation of a simpler form if the coefficients of the equation are assumed in the form of polynomials which is usually done in practice.

If the coefficients $a_i(x)$ are assumed in the form of polynomials, the Eq. (3.3) takes the form

$$\Big(\alpha_0 + \sum_{k=1}^{n} \alpha_k x^k\Big) X^{IV} + \Big(\beta_0 + \sum_{k=1}^{n} \beta_k x^k\Big) X''' + \Big(\gamma_0 + \sum_{k=1}^{n} \gamma_k x^k\Big) X'' +$$
$$+ \Big(\nu_0 + \sum_{k=1}^{n} \nu_k x^k\Big) X' + \Big(\mu_0 + \sum_{k=1}^{n} \mu_k x^k\Big) X = 0 \tag{3.13}$$

where the constant coefficients are complex, in general. In order to preserve the polynomial form of the coefficients the equation is not divided by the coefficient of X^{IV} as was done in the former method.

Applying the Laplace transformation to (3.13) and bearing in mind that

$$L\{x^m X(x)\} = (-1)^m \frac{\mathrm{d}^m \overline{X}(p)}{\mathrm{d}p^m} \tag{3.14}$$

where

$$L\{y\} = \int_0^\infty e^{-px} y(x) \mathrm{d}x \tag{3.15}$$

the Eq. (3.13) may be reduced (by making use of the initial parameters (3.6)) to the form

$$\begin{aligned} &a_0(\overline{X}p^4-p^3A-p^2B)+\sum_{k=1}^{n}(-1)^k a_k \frac{\mathrm{d}^k}{\mathrm{d}p^k}(\overline{X}p^4-p^3A-p^2B)+\\ &+\beta_0(\overline{X}p^3-p^2A-pB)+\sum_{k=1}^{n}(-1)^k \beta_k \frac{\mathrm{d}^k}{\mathrm{d}p^k}(\overline{X}p^3-p^2A-pB)+\\ &+\gamma_0(\overline{X}p^2-pA-B)+\sum_{k=1}^{n}(-1)^k \gamma_k \frac{\mathrm{d}^k}{\mathrm{d}p^k}(\overline{X}p^2-pA-B)+\\ &+\nu_0(\overline{X}p-A)+\sum_{k=1}^{n}(-1)^k \nu_k \frac{\mathrm{d}^k}{\mathrm{d}p^k}(Xp-A)+\mu_0\overline{X}+\sum_{k=1}^{n}\mu_k \frac{\mathrm{d}^k\overline{X}}{\mathrm{d}p^k}=0 \end{aligned} \tag{3.16}$$

or, after rearrangement,

$$a_n \frac{\mathrm{d}^n\overline{X}}{\mathrm{d}p_n}+a_{n-1}\frac{\mathrm{d}^{n-1}\overline{X}}{\mathrm{d}p^{n-1}}+\cdots+a_0\overline{X}=H(p) \tag{3.17}$$

Hence

$$\overline{X}=\frac{H(p)}{a_0(p)}-\frac{a_n(p)}{a_0(p)}\overline{X}^n-\cdots-\frac{a_1(p)}{a_0(p)}X' \tag{3.18}$$

$a_0, \dots, a_j$ are, in general, fourth order polynomials in p depending additionally of the initial parameters A, B. Assuming, that a_0 has single roots, the free term and the coefficients of the derivatives in (3.18) may be expanded thus

$$\frac{H(p)}{a_0(p)}=\sum_{r=1}^{4}\frac{kr}{p-pr}\,; \qquad -\frac{a_j(p)}{a_0(p)}=b_{j0}+\sum_{r=1}^{4}b_{jr}\frac{1}{p-p_r} \tag{3.19}$$

Since, due the transformation law of a convolution, we have

$$L^{-1}\left\{-\frac{a_j(p)}{a_0(p)}\frac{\mathrm{d}^j\bar{X}(p)}{\mathrm{d}p^j}\right\}=L^{-1}\left\{\left(b_{j0}+\sum_{r=1}^{4}\frac{b_{jr}}{p-p_r}\right)\frac{\mathrm{d}^j\bar{X}(p)}{\mathrm{d}p^j}\right\}$$

$$=\sum_{r=1}^{4}b_{jr}\int_0^x \xi^j X(\xi)e^{P_r(x-\xi)}\,\mathrm{d}\xi+(-1)^j x^j X(x)b_{j0} \quad (3.20)$$

therefore the inverse transform of the Eq. (3.18) can be represented directly in the form

$$X(x)=f(x)+\int_0^x K(x,\xi)X(\xi)\,\mathrm{d}\xi \quad (3.21)$$

where

$$f(x)=\frac{\sum\limits_{r=1}^{4}k_r e^{P_r X}}{1-\sum\limits_{j=0}^{n}(-1)^j b_{j0}x^j} \quad (3.22)$$

$$K(x,\xi)=\sum_{j=1}^{n}\sum_{r=1}^{4}\frac{b_{jr}\xi^j e^{P_r(x-\xi)}}{1-\sum\limits_{j=1}^{n}(-1)^j b_{j0}x} \quad (3.23)$$

The Eq. (3.21) is the sought for Volterra integral equation of the second kind, of which the solution has the form

$$X(x)=f(x)+\sum_{m=1}^{\infty}\int_0^x K_m(x,s)f(s)\,\mathrm{d}s \quad (3.24)$$

where

$$K_m(x,s)=\int_0^l K(x,\xi)K_{m-1}(\xi,s)\,\mathrm{d}\xi \quad (3.25)$$

If

$$|K(x-\xi)|<M \qquad \text{and} \qquad \int_0^1 M|f(s)|\,\mathrm{d}s=\varkappa$$

the absolute error of the n-th approximation to the solution (3.24) will be obtained from the equation

$$\Delta X\leqslant\left|\varkappa e^{Ml}-\sum_{m=1}^{n}\frac{(Ml)^{m-1}}{(m-1)!}\right| \quad (3.26)$$

After determination the function $X(x)$ either from (3.1) by means of (3.8) or from (3.21) by means of (3.24) we must arrange the obtained solution in relation to A and B.

$$X(x) = AR_1(x,U,\omega)+BR_2(x,U,\omega) \tag{3.27}$$

Then, using the two remaining boundary conditions (3.5) we obtain the characteristic determinant of the problem

$$\begin{vmatrix} R_1''(l,U,\omega)\,; & R_2''(l,U,\omega) \\ R_1'''(l,U,\omega)\,; & R_2'''(l,U,\omega) \end{vmatrix} = 0 \tag{3.28}$$

Hence, by the method described above, assuming ω in the form (3.4) the critical velocity is calculated.

In the case where the coefficients of the equations cannot be expressed by a single relation for the entire rocket, the Eq. (3.3) must be split up into two parts, the contact conditions being used.

The case of the Eq. (2.14) is similar. We shall describe in brief the solution method of the Eq. (2.14).

With a certain change of the contact conditions it will be preserved in the case of necessity of splitting up the equation (3.3) into two.

Assuming (3.1) and rearranging, the Eq. (2.14) cane be written thus

$$X^{\mathrm{IV}}+a_1(x)X'''+a_2(x)X''+a_3(x)X'+a_4(x)X = [b_1X(x_0)+b_2X'(x_0)]\,\delta(x-x_0) \tag{3.29}$$

This equation will be split up into two

$$\begin{aligned} &X^{\mathrm{IV}}+a_1(x)X'''+a_2(x)X''+a_3(x)X'+a_4(x)X_{x\leqslant x_0} = 0 \\ &X_1^{\mathrm{IV}}+b_1(x_1)X_1'''+b_2(x_1)X_1''+b_3(x_1)X_1'+b_4(x_1)X_{1x\geqslant x_0} = 0 \end{aligned} \tag{3.30}$$

where

$$x_1 = l-x.$$

The boundary conditions are

$$X''(0) = X'''(0) = 0\,; \qquad X_1''(0) = X_1'''(0) = 0 \tag{3.31}$$

$$\begin{aligned} [EJX'']'_{x=x_0}+[EJX_1'']'_{x_1=l-x_0} &= b_1X(x_0)+b_2[X']_{x=x_0} \\ [EJX'']_{x=x_0} &= [EJX_1'']_{x_1=l-x_0} \\ [X']_{x=x_0}+[X_1']_{x_1=l-x_0} &= 0 \\ X(x_0) &= X_1(l-x_0) \end{aligned} \tag{3.32}$$

If, according to the above method, the equations (3.30) are reduced, each one in its own coordinate system, to Volterra integral equations, the method of initial parameters being used, two parameters A and B (according to notations analogous to (3.6)) will remain undetermined in the equation for x after using the conditions (3.31), as well as the parameters

A_1 and B_1 in the equation for x_1. Solving both Volterra equations and rearranging in relation to A, B, A_1, B_1, we can obtain the characteristic determinant, on the basis of the conditions (3.32). Then, the solution is continued as before. In the case where the coefficients in the Eq. (3.3) have different expressions in different parts of the fuselage, the solution is analogous, except that b_1 and b_2 in the boundary conditions (3.32) should be made zero.

The above solution is a general theoretical solution of the problem and enables us to obtain approximate solutions with any degree of accuracy. However, in practical computations it requires a considerable labour and is useful only if high speed computers are available.

Much simpler solutions, although without the possibility of appraisal of the accuracy of results, may be obtained by means of the approximate method to be considered below.

4. *Approximate Method*

The approximate method for solving the Eq. (2.9) is based on the modified Galerkin method.

The solution is assumed in the form

$$y = a_0(t)y_0 + a_1(t)y_1 + \dots + a_n(t)y_n \tag{4.1}$$

where

$a_i(t)$ — are functions of time not yet determined

$i = 0, 1, \dots, n$

$y_0 = 1$ — characterises the rigid displacement of the rocket

$y_1 = (x - x_{cg})$ — characterises the rigid rotation of the rocket about the gravity centre x_{cg}

$y_j(x)$ — eigenfunctions of vibration for the rocket treated

$j = 2, 3, \dots, n$ as a free beam with boundary conditions

$$y''(0) = y'''(0) = y''(l) = y'''(l) = 0$$

where y_2 is the first harmonic.

The equation of free vibration of a beam

$$\frac{\partial^2}{\partial x^2} EJ \frac{\partial^2 y_m}{\partial x^2} = -A(x)\varrho^*(x)\omega_m^2 y_m \tag{4.2}$$

$$m = 2, \dots, n$$

enables us to determine the eigenvalues ω_m.

The eigenfunctions satisfy the orthogonality condition

$$\int_0^l A(x)\varrho^*(x) y_i y_j \,dx \begin{matrix} = 0 & i \neq j \\ \neq 0 & i = j, \; i,j = 0, 1, \dots, n, \end{matrix} \tag{4.3}$$

Substituting (4.1) in (2.14) and using (4.2) we obtain, the damping being disregarded for simplicity,

$$A\varrho^*[a_2\omega_2^2y_2+a_3\omega_3^2y_3+...+a_n\omega_n^2y_n] = -A(\varrho_0+\varrho^*)(a_0''y_0+ +a_1''y_1+...+a_n''y_n)-2\varrho_0UA(a_0'y_0'+a_1'y_1'+...+a_n'y_n')+ -\varrho_0AU^2(a_0y_0''+a_1y_1''+...+a_ny_n'')-\varrho_0U\frac{\mathrm{d}A}{\mathrm{d}x}(Q_0'y_0+a_1'y_1+ +...+a_n'y_n)-\varrho_0U^2\frac{\mathrm{d}A}{\mathrm{d}x}(a_0y_0'+a_1y_1'+...+a_ny_n')+\delta(x-x_0)F_t \quad (4.4)$$

The prime denoting the time derivative of $a_i(t)$ or x — derivative of y.

Let us apply to the Eq. (4.4) the modified Galerkin's orthogonality condition. Multiplying (4.4) by each succesive $y_i\,(i=0,1,...,n)$, integrating from 0 to l and remembering that the integral of the Dirac function multiplied by a given function gives its particular values, we obtain, use being made of the orthogonality of the eigenfunctions (4.3), the following system of equations.

$$0=-a_0''c_0-\sum_{j=0}^{n}[\mathrm{d}_{0j}+\alpha_2(y_jy_0)_{x=x_0}+\alpha_5(y_j'y_0)_{x=x_0}]a_j''-\sum_{j=0}^{n}[U(e_{0j}+g_{0j})+ +\alpha_1(y_jy_0)_{x=x_0}+\alpha_4(y_j'y_0)_{x=x_0}]a_j'-\sum_{j=0}^{n}[U^2(f_{0j}+h_{0j})+\alpha_3(y_j'y_0)_{x=x_0}]a_j,$$

$$0=-a_1''c_1-\sum_{j=0}^{n}[\mathrm{d}_{1j}+\alpha_2(y_jy_1)_{x=x_0}+\alpha_5(y_j'y_1)_{x=x_0}]a_j''-\sum_{j=0}^{n}[U(e_{1j}+g_{1j})+ +\alpha_1(y_jy_1)_{x=x_0}+\alpha_4(y_j'y_1)_{x=x_0}]a_j'-\sum_{j=0}^{n}[U^2(f_{1j}+h_{1j})+\alpha_3(y_j'y_1)_{x=x_0}]a_j,$$

$$a_2\omega_2^2c_2=-a_2''c_2-\sum_{j=0}^{n}[\mathrm{d}_{2j}+\alpha_2(y_jy_2)_{x=x_0}+ +\alpha_5(y_j'y_2)_{x=x_0}]a_j''-\sum_{j=0}^{n}[U(e_{2j}+g_{2j})+\alpha_1(y_jy_2)_{x=x_0}+\alpha_4(y_j'y_2)_{x=x_0}]a_j'- -\sum_{j=0}^{n}[U^2(f_{2j}+h_{2j})+\alpha_3(y_j'y_2)_{x=x_0}]a_j,$$

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$$a_n\omega_n^2c_n=-a_n''c_n-\sum_{j=0}^{n}[\mathrm{d}_{nj}+\alpha_2(y_jy_n)_{x=x_0}+ +\alpha_5(y_j'y_n)_{x=x_0}]a_j''-\sum_{j=0}^{n}[U(e_{nj}+g_{nj})+\alpha_1(y_jy_n)_{x=x_0}+\alpha_4(y_j'y_n)_{x=x_0}]a_j'+ -\sum_{j=0}^{n}[U^2(f_{nj}+h_{nj})+\alpha_3(y_j'y_n)_{x=x_0}]a_j \quad (4.5)$$

where

$$c_i = \int_0^l A\varrho^* y_i^2 \, dx$$

$$d_{ij} = \varrho_0 \int_0^l A y_i y_j \, dx$$

$$e_{ij} = 2\varrho_0 \int_0^l A y_i y_j' \, dx$$

$$f_{ij} = \varrho_0 \int_0^l A y_i y_j'' \, dx$$

$$g_{ij} = \varrho_0 \int_0^l \frac{dA}{dx} y_i y_j \, dx$$

$$h_{ij} = \varrho_0 \int_0^l \frac{dA}{dx} y_i y_j' \, dx \tag{4.6}$$

The solution of the Eq. (4.5.) is sought in the form

$$a_n = B_n e^{\lambda t} \tag{4.7}$$

then, we obtain from (4.5)

$$\begin{aligned} O = & -\lambda^2 \{c_0 B_0 + \sum_{j=0}^{n} [d_{0j} + \alpha_2 (y_j y_0)_{x=x_0} + \alpha_5 (y_j' y_0)_{x=x_0}] B_j\} - \lambda \sum_{j=0}^{n} [U(e_{0j} + \\ & + g_{0j}) + \alpha_1 (y_j y_0)_{x=x_0} + \alpha_4 (y_j' y_0)_{x=x_0}] B_j - \sum_{j=0}^{n} [U^2 (f_{0j} + h_{0j}) + \alpha_3 (y_j' y_0)_{x=x_0}] B_j, \\ 0 = & -\lambda^2 \{c_1 B_1 + \sum_{j=0}^{n} [d_{1j} + \alpha_2 (y_j y_1)_{x=x_0} + \alpha_5 (y_j' y_1)_{x=x_0}] B_j\} - \lambda \sum_{j=0}^{n} [U(e_{1j} + \\ & + g_{1j}) + \alpha_1 (y_j y_1)_{x=x_0} + \alpha_4 (y_j' y_1)_{x=x_0}] B_j - \sum_{j=0}^{n} [U^2 (f_{1j} + h_{1j}) + \alpha_3 (y_j' y_1)_{x=x_0}] B_j \end{aligned} \tag{4.8}$$

$$\begin{aligned} B_2 \omega_2^2 c_2 = & -\lambda^2 \{c_2 B_2 + \sum_{j=0}^{n} [d_{2j} + \alpha_2 (y_j y_2)_{x=x_0} + \\ & + \alpha_5 (y_j' y_2)_{x=x_0}] B\} - \lambda \sum_{j=0}^{n} [U(e_{2j} + g_{2j}) + \alpha_1 (y_j y_2)_{x=x_0} + \alpha_4 (y_j' y_2)_{x=x_0}] B_j + \\ & - \sum_{j=0}^{n} [U^2 (f_{2j} + h_{2j}) + \alpha_3 (y_j' y_2)_{x=x_0}] B, \end{aligned}$$

. . . .

. . . .

$$B_n\omega_n^2c_n = -\lambda^2\{c_nB_n+\sum_{j=0}^{n}[d_{nj}+a_2(y_jy_n)_{x=x_0}+$$

$$+a_5(y'_jy_n)_{x=x_0}]B_j\}-\lambda\sum_{j=0}^{n}[U(e_{nj}+g_{nj})+a_1(y_jy_n)_{x=x_0}+a_4(y'_jy_n)_{x=x_0}]B_j+$$

$$-\sum_{j=0}^{n}[U^2(f_{nj}+h_{nj})+a_3(y'_jy_n)_{x=x_0}]B_j \qquad (4.8)$$

Assuming an appropriate finite number of terms $n \geqslant 3$ (for a given degree of accuracy) and solving the system (4.8) for $B_j(j = 0, 1, ..., n)$ the characteristic equation of the $2n$-th order enabling the determination of the value of the roots in function of velocity is obtained.

The last value of velocity with $Re\lambda$ changing sign from negative to positive is the flutter velocity.

5. *Examples*

Two numerical examples will be given below. The first—for a deformable rocket with tail, the other—for a two-stage rocket with tail, composed of two rigid parts connected elastically. In both cases the flutter velocity is found. The approximate method of §4 is used, with the purpose of showing the order of magnitude of the flutter velocities.

The accurate solution can also be obtained, of course. It requires a considerable labour, however. The verification of the approximate method by means of the accurate method will be done separately.

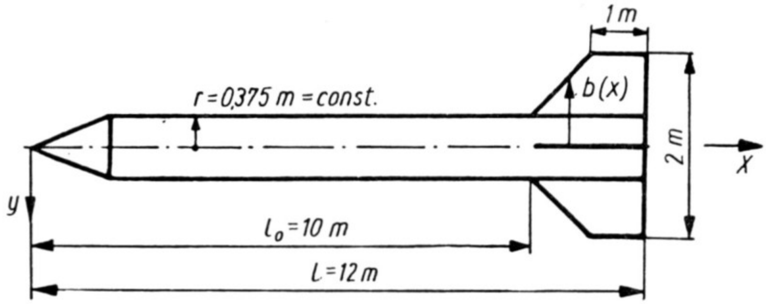

FIG. 2.

Example 1—Assuming the type of the rocket as shown at Fig. 2 the following numerical data are assumed

$$r = 0.375 \text{ m}$$
$$A = \text{const} = \pi r^2$$
$$\varrho_0 = 0.1 \text{ kG sec}^2/\text{m}^4$$

$$\begin{aligned} \varrho^* &= 100 \text{ kG sec}^2/\text{m}^4 \\ EJ &= 3\cdot 10^6 \text{ kGm}^2 \\ l &= 12 \text{ m} \end{aligned} \tag{5.1}$$

For the simplified equation (4.2) the eigenfunctions have the form

$$y_i(x) = \frac{\sin k_i l - \operatorname{sh} k_i l}{\operatorname{ch} k_i l - \cos k_i l}(\cos k_i x + \operatorname{ch} k_i x) + \sin k_i x + \operatorname{sh} k_i x \tag{5.2}$$

where $k_2 l = 4.73$; $k_3 l = 7.853$.
Assuming for simplicity that $x_0 = l$
and taking $n = 3$ we obtain, after some computation, the following value of critical velocity

$$U_{cr} = 1945 \text{ m/sec} \tag{5.3}$$

Example 2—The rocket under consideration is a two-stage rocket composed of two rigid stages elastically connected, the coefficient of elastic joint being K. The numerical data will be those of the *Example* 1. The notations and the scheme of the rocket are shown at Fig. 3.

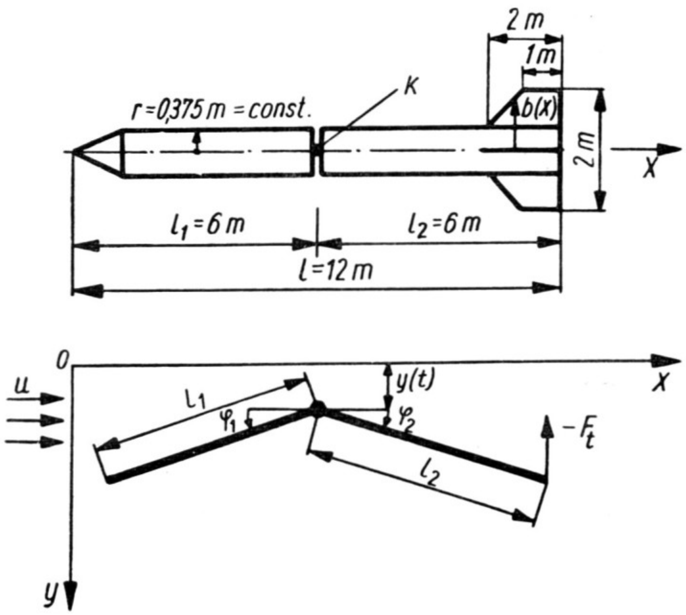

FIG. 3.

The equations of vibration take, by the Eq. (3.14), the form

$$\varkappa_2 l-\alpha_2)y''+\left(\frac{\varkappa_1}{2}-\alpha_1\right)y'+\varkappa_2\frac{l_1^2}{2}\varphi_1''-\varkappa_1\frac{l_1}{2}\varphi_1'+\left(\frac{l_2^2}{2}\varkappa_2-\alpha_2 l_2-\alpha_5\right)\varphi_2''+$$
$$+(\varkappa_1 l_2-\alpha_1 l_2-\alpha_4)\varphi_2'+(\varrho_0 U^2 A-\alpha_3)\varphi_2 = 0$$

$$\varkappa_2 \frac{l_1^2}{2} y'' + \varkappa_1 \frac{l_1}{2} y' + \varkappa_2 \frac{l_1^3}{3} \varphi_1'' + \varkappa_2 \frac{l_1^3}{3} \varphi_1'' + (K - \varrho_0 U^2 A l_1) \varphi_1 + k\varphi_2 = 0$$

$$\varkappa_2 \frac{l^2 - l_1^2}{2} y'' + \varkappa_1 \frac{l_2}{2} y^1 \frac{l_1^2 l_2}{2} \varkappa_2 \varphi_1'' - \frac{\varkappa_1 l_1 l_2}{2} \varphi_1' - k\varphi_1 +$$

$$+ \varkappa_2 \left(\frac{l^3 - l_1^3}{6} - \frac{l l_1 l_2}{2} \right) \varphi_2'' + \varkappa_1 \frac{l_2^2}{2} \varphi_2' + (\partial U^2 A l_2 - K) \varphi_2 = 0$$

where

$$\varkappa_1 = 2\varrho_0 U A, \quad \varkappa_2 = (\varrho_0 + \varrho^*) A \tag{5.4}$$

The accurate solution of the characteristic equation for the system of equations (5.4) gives, under the assumption

$$K = 1 \cdot 10^5 \text{ kGm}$$

the following critical value of velocity $U_{cr} = 998$ m/sec.

For other rigidities K the flutter velocity varies proportionally to $\sqrt{K}$. If $K \to \infty$, there is no critical velocity.

6. *Conclusion*

Solutions based on approximate methods for integrating the equations of the problem are presented. These methods do not enable accurate appraisal of the error. This may be obtained by means of the general solution method described in § 3. Examples verifying approximate solutions are not given in the present paper and will be the object of separate considerations.

It should be stated, in addition, that the aerodynamic part of the problem is treated with a far going simplification. In the next stage of the work this problem will be dealt with in greater detail, in particular from the viewpoint of non-linear effects, interference between the fuselage and the tail and nonsteady state flow.

The next problem suggested by the present considerations is that of composite types of rocket vibration that is flexural- longitudinal- torsional vibration and a motion along a curvilinear path.

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